1 Introduction

Research in droplet impact on solid surfaces has received extensive attention since the earliest work of Worthington [1]. For the past few decades, the interest in this subject has been rising due to a number of emerging applications based on controlled deposition of droplets such as inkjet printing, spray coating, and soldering. In these applications, one of the most important process parameters is the maximum spread of the droplet $D_{\text{max}}=D_{\text{max}}/D_0$, where $D_{\text{max}}$ is the measured maximum spread diameter and $D_0$ is the initial diameter of the droplet. The maximum spread is determined by droplet initial diameter ($D_0$), impact velocity ($U_0$), surface tension ($\sigma$), viscosity ($\mu$), and wettability of the solid surface (represented by the equilibrium contact angle $\theta_e$). These parameters can be organized into Reynolds and Weber numbers given by

$$\text{We} = \frac{\rho D_0 U_0^2}{\sigma}$$

$$\text{Re} = \frac{\rho D_0 U_0}{\mu}$$

Numerous studies have been carried out to examine the maximum spread $D_m$. A few analytical models were proposed to resolve the transient spreading of droplets [2–6]. Efforts have been made to develop analytical equations to predict the maximum spread of droplets on solid surfaces. Asai et al. [7] and Scheller and Bousfield [8] correlated $D_m$ with Weber and Reynolds numbers of the droplets. A few other equations were based on the energy conservation of the initial and maximum spread states in which $\text{Re}$, $\text{We}$, $\Theta$, and $\theta_e$ are included. Some studies proposed analytical equations and compared with their experimental results [9–12] or with their numerical results [10,13]. Some other studies are entirely theoretical [14,15]. A number of notable experimental studies were also published [16–20].

Most of the analytical equations available in literature have been used to predict the maximum spread of droplets impacting with high velocities. In case of high impact velocities, spreading is driven by inertia and capillary effect becomes less significant [21].

However, limitation arises when the previous equations are applied to droplet impact with low velocities, especially when inertia is comparable to or less than capillarity ($\text{We} \ll 10$), when inertia is comparable to or less than viscous effect ($\text{Re} \ll 10$), and when the capillary and viscous effects are comparable ($\text{Oh} \sim O(1)$). Here, $\text{Oh}$ is the Ohnesorge number given by

$$\text{Oh} = \sqrt{\frac{\text{We}}{\text{Re}}}$$

Additionally, for low impact velocities, the wettability of the solid surface becomes significant and the change of potential energy of the drop-surface system has to be treated accurately, especially for small values of $\theta_e$.

A few previous equations are plotted in Fig. 1. Figure 1(a) shows maximum spread for small Weber numbers $\text{We} < 10$, $\text{Oh} = 1$, and $\theta_e = 30$ deg. The equation for high viscosity in Park et al. [12] is used. The advancing contact angle used in Pasandideh-Fard et al. [10] is replaced with equilibrium contact angle. In Fig. 1(a), all equations predict that $D_m$ decreases with an increase in $\text{We}$, showing a physically incorrect trend. Additionally, different equations predict different values of $D_m$ for $\text{We}=0$. Figure 1(a) can also be plotted for $\theta_e = 180$ deg, which is not shown here. If we consider a droplet with zero impact velocity on a hydrophobic surface ($\Theta_e = 180$ deg), most of the equations predict different nonzero values. The only exception is Park et al. [12], which predicts $D_m = 0$ for $\text{We}=0$. The potential energy of drop-surface system was assumed to be completely dissipated at maximum spread, which actually forces the maximum spread of a spontaneously spreading droplet to its equilibrium spread. Our analysis will show that this assumption is inaccurate. Figure 1(b) shows the previous equations as a function of $\theta_e$, for $\text{We}=1$ and $\text{Oh} = 1$. It is expected that $D_m$ increases with a decrease in $\theta_e$ and that $D_m \to \infty$ when $\theta_e \to 0$ deg. Clearly, none of the equations gives correct predictions in Fig. 1(b).

Droplet impact with low Weber and Reynolds numbers has not been covered by previous models. The primary objective of this study is to develop a theoretical model that is capable of predicting the maximum spread of droplets on solid surfaces with zero or low Weber and Reynolds numbers. The droplet shape during spreading is assumed to be a spherical cap. Surface tension, inertia, viscosity, and wettability will be treated carefully since they are comparably important in determining the maximum spread.

This theoretical study proposes an analytical model to predict the maximum spread of single droplets on solid surfaces with zero or low Weber and Reynolds numbers. The spreading droplet is assumed as a spherical cap considering low impact velocities. Three spreading states are considered, which include equilibrium spread, maximum spontaneous spread, and maximum spread. Energy conservation is applied to the droplet as a control volume. The model equation contains two viscous dissipation terms, each of which has a defined coefficient. One term is for viscous dissipation in spontaneous spreading and the other one is for viscous dissipation of the initial kinetic energy of the droplet. The new model satisfies the fundamental physics of drop-surface interaction and can be used for droplets impacting on solid surfaces with or without initial kinetic energy. [DOI: 10.1115/1.4001695]
2 New Model

A new model for predicting maximum spread of droplets on solid surfaces will be developed in this section. There are two assumptions to make. First, the solid surface is assumed to be perfectly smooth. Wetting on rough surfaces [22] is beyond the scope of the present work since the effect of surface roughness on viscous dissipation is difficult to determine analytically. Second, the effect of gravity is neglected. This assumption is valid for droplets with small Bond numbers \( \text{Bo} = \rho g D_0^2 / \sigma < 0.1 \). Otherwise, complex geometric relation is needed to track the center of mass during droplet spreading, which is not included in our model.

All the terms and variables that appear in the equations have been nondimensionalized. Energy and work terms have been nondimensionalized by \( \pi D_0^3 \sigma \). All lengths have been nondimensionalized by \( D_0 \), times by \( D_0 / U_0 \), velocities by \( U_0 \), areas by \( \pi D_0^2 \), and volumes by \( \pi D_0^3 \). All surface tension coefficients have been nondimensionalized by the liquid-air surface tension \( \sigma \).

In previous models, the droplet shape has been assumed to be either cylindrical disk [5,9,10,13,15] or spherical cap [5,12]. For zero or low impact velocities, it is reasonable to assume that the droplet maintains a spherical cap shape during the spreading process. The following geometric relations of a spherical cap will be used:

\[
\begin{align*}
A(d) &= \sqrt[3]{\frac{1}{2} \left( \frac{1}{64} + \frac{d^6}{64} \right) + \frac{1}{4} \left( \frac{1}{64} + \frac{d^6}{64} \right) + \sqrt{\frac{1}{2} \left( \frac{1}{64} + \frac{d^6}{64} \right) - \frac{1}{4} \left( \frac{1}{64} + \frac{d^6}{64} \right) - \frac{d^2}{4}}} \\
\theta_c &= \sin^{-1} \left( \frac{2}{\sqrt{1 - \cos \theta}} \right) \left( \frac{1}{4} + \frac{2 \cos \theta}{} \right) \frac{1}{4}
\end{align*}
\]  

Equation (4) shows liquid-air interface area \( A \) as a function of base diameter \( d \). Equation (5) expresses droplet height \( h \) as a function of base diameter \( d \). Equation (6) gives the base diameter \( d \) and base radius \( r \) as a function of contact angle \( \theta \).

Before starting the theoretical analysis, we first consider four particular spreading states of the droplet as shown in Fig. 2. Initially, the droplet is a spheroid body with its nondimensional diameter being unity (state 1 in Fig. 2). Assuming the droplet as a spherical cap during spreading, the evolution of the spherical cap will pass an equilibrium shape (state 2 in Fig. 2), maximum spontaneous spread shape (state 3 in Fig. 2) and maximum spread shape (state 4 in Fig. 2). The equilibrium shape, which can be characterized by equilibrium spread \( D_e \) or its equilibrium contact angle \( \theta_e \). In the case of zero impact velocity, the droplet has a maximum spontaneous spread \( D_s \) and height \( H_s \). If the droplet initially has kinetic energy, the droplet has maximum spread \( D_m \) and height \( H_m \). Comparing states 2–4, it requires \( D_c \leq D_s \leq D_m \).

To model the maximum spread of droplets on solid surfaces, we choose the droplet as a control volume and assume the control
volume to have zero kinetic energy at maximum spread. Additionally, the gravity effect is neglected. Applying energy conservation to the control volume, we require
\[
E_k^i + (W^W - W^\theta - E^{KV})_m - E^{SV} = 0
\]  
(7)

The subscripts \(i\) and \(m\) represent the initial state (state 1 in Fig. 2) and maximum spread (state 4 in Fig. 2), respectively. \(E_k^i\) is the initial kinetic energy of the droplet. \(W^W\) is the work done by radial forces applied from the solid surface to contact line. \(W^\theta\) is the work done by the surface tension force applied from the fluid to the contact line. There are two viscous dissipation terms in Eq. (7): \(E^{KV}\) represents a portion of the initial kinetic energy that is viscously dissipated and \(E^{SV}\) is part of potential energy that is viscously dissipated.

The initial kinetic energy of the droplet is
\[
E_k^i = \frac{W_e}{12}
\]  
(8)
The work done by the solid surface to the contact line is
\[
W^W = \int_{0}^{D/2} (\gamma_{se} - \gamma_{sl})2dr
\]  
(9)
where \(D\) is the spread diameter of the droplet. The solid-air and solid-liquid surface tensions \(\gamma_{se}\) and \(\gamma_{sl}\) are nondimensional, which have been normalized by liquid-air surface tension \(\sigma\). These two surface tensions meet the following relation, known as Young’s equation [23]:
\[
\gamma_{se} = \cos \theta_e + \gamma_{sl}
\]  
(10)
Inserting Eq. (10) into Eq. (9) and carrying out integration yield
\[
W^W = \frac{D^2}{4} \cos \theta_e
\]  
(11)
During spreading, the fluid applies surface tension force to the contact line. The resultant work can be expressed by
\[
W^\theta = \int_{0}^{D/2} \cos \theta d2dr
\]  
(12)
Combining Eq. (12) with Eqs. (4) and (6) gives
\[
W^\theta = A(D) - 1
\]  
(13)
Equation (13) shows that \(W^\theta\) is the change of liquid-air surface energy of the droplet. This indicates the work done by the fluid to contact line and the change in liquid-air surface energy should not co-exist in the energy conservation equation.

Considering the potential energy of a closed system that contains the droplet and the solid surface, we combine Eqs. (11) and (13) into
\[
P(D) = W^W - W^\theta = \frac{D^2}{4} \cos \theta_e - A(D) + 1
\]  
(14)
Here, \(P\) represents the amount of potential energy that is released during spreading. Figure 3 shows \(P\) as a function of \(D\) for \(P \geq 0\). When the droplet reaches its equilibrium shape, \(P\) is maximum, i.e., \(P(D_e)\), where \(D_e = d(\theta_e)\) is the equilibrium spread (see the dotted lines in Fig. 3).

For a droplet with zero initial kinetic energy, Eq. (7) reduces to
\[
P(D) - E^{SV} = 0
\]  
(15)
where \(D_s\) is the maximum spread of the droplet during spontaneous spreading (corresponding to state 3 in Fig. 2). From Fig. 3, \(D_s = D_e\) and \(P(D_s) = P(D_e)\). Hence, we define
\[
E^{SV} = C_s P(D_e)
\]  
(16)
The coefficient \(C_s\), termed as potential energy dissipation coefficient, is given by

![Fig. 3 The amount of potential energy that is released during spontaneous spreading (Eq. (14)). The dotted lines show the maxima of \(P\) at equilibrium spread.](image)

\[
C_s = \frac{P(D_s)}{P(D_e)}
\]  
(17)
which reflects the viscous effect during spontaneous spreading and should satisfy \(0 \leq C_s \leq 1\). For inviscid droplets, \(C_s = 0\) and \(P(D_e) = 0\). For highly viscous droplets, \(C_s = 1\), \(P(D_e) = P(D_s)\), and \(D_s = D_e\). For \(\theta_e = 0\) deg, \(D_s = D_e \rightarrow \infty\), \(P(D_e) = P(D_s) = 1\). Therefore, \(C_s = 1\) for \(\theta_e = 0\) deg. \(C_s\) can be determined either experimentally or numerically. Carrying out numerical modeling or experimental test on a spontaneously spreading droplet, the measured maximum spread \(D_s\) and equilibrium spread \(D_e\) are then put into Eq. (17) to calculate \(C_s\). Figure 4 shows the maximum spontaneous spread \(D_s\) as a function of \(C_s\).

A portion of initial kinetic energy is viscously dissipated, which is expressed by
\[
E^{KV} = \frac{W_e}{Re} \int_{0}^{r_e} \int_{0}^{\Omega} \Phi d\Omega d\tau
\]  
(18)
where \(\Omega\) is the volume of viscous fluid normalized by \(\pi D_0^3\). Here, \(\Phi\) is the dissipation function, which has been nondimensionalized by \(\mu U_0^2/D_0^3\) and in cylindrical coordinates is expressed by

![Fig. 4 Maximum spread of a spontaneously spreading droplet as a function of \(C_s\)](image)
\[
\Phi = 2 \left( \frac{\partial u_t}{\partial r} \right)^2 + \left( \frac{u_t}{r} \right)^2 + \left( \frac{\partial u_r}{\partial \tau} \right)^2 + \frac{1}{2} \left( \frac{\partial u_r}{\partial \tau} + \frac{\partial u_t}{\partial \tau} \right)^2 \right)
\]

Chandra and Avedisian [9] simplified Eq. (19) to
\[
\Phi \sim \left( \frac{\partial u_t}{\partial \tau} \right)^2 \sim \left( \frac{1}{\delta} \right)^2
\]

where the velocity gradient is approximated as the droplet impact velocity (being unity in nondimensional form) divided by the nondimensional thickness of viscous boundary layer (\(\delta\)). The volume \(\Omega\) can be simplified to be
\[
\Omega \sim \frac{d^2}{4 \delta}
\]

We assume that during spreading the top of the droplet maintains a constant velocity equal to the droplet impact velocity. Hence,
\[
\frac{dh}{d\tau} \sim -1
\]

where \(\tau\) is nondimensional time. Equation (22) can be integrated to obtain
\[
\tau \sim 1 - h
\]

Thus, Eq. (18) can be changed to
\[
E^{KV} \sim - \frac{W e}{Re} \frac{1}{4 \delta} \int_{h_i}^{H} d^2 dh
\]

where \(H\) is the droplet height at \(\tau=t\), i.e., \(H=h(D)\). In Eq. (24), we have also assumed \(\delta\) to be constant during spreading. If we define the droplet height at maximum spontaneous spread as \(H_s\), i.e., \(H_s=h(D_s)\), Eq. (24) can be rewritten as
\[
E^{KV} \sim \frac{W e}{Re} \frac{1}{4 \delta} \int_{h_i}^{H_s} d^2 dh + \frac{W e}{Re} \frac{1}{4 \delta} \int_{H_s}^{H} d^2 dh
\]

Since \(E^{KV}\) is defined as the viscous dissipation only to the initial kinetic energy, the first term on the right-hand side must not be larger than the initial kinetic energy. Hence, we introduce \(C_K\) termed as kinetic energy dissipation coefficient and that term can be expressed as
\[
E^{KV} \sim \frac{W e}{Re} \frac{1}{4 \delta} \int_{h_i}^{H_s} d^2 dh = C_K E^K
\]

where \(0 \leq C_K \leq 1\). There will be further discussion on this coefficient later on. Based on the stagnation point flow, Pasandideh-Fard et al. [10] proposed
\[
\delta = 2 \sqrt{Re}
\]

Inserting Eqs. (26) and (27) into Eq. (25), we get
\[
E^{KV} \sim \frac{W e}{12} \left( C_K + \frac{3}{2 \sqrt{Re}} \int_{h_i}^{H_s} d^2 dh \right)
\]

Substituting Eqs. (8), (14), (16), and (28) into Eq. (7) gives
\[
\frac{W e}{12} \left[ P(D) - \frac{W e}{12} \left( C_K + \frac{3}{2 \sqrt{Re}} \int_{h_i}^{H_s} d^2 dh \right) \right] = C_K P(D_s) = 0
\]

Equation (29) can be rearranged into
\[
\frac{W e}{12} \left( 1 - C_K - \frac{3}{2 \sqrt{Re}} \int_{h_i}^{H_s} d^2 dh \right) = C_2 P(D_s) - P(D_m)
\]

The droplet height at maximum spread \(H_m\) can be determined from \(H_m=h(D_m)\), and the function \(P\) is given by Eq. (14). The left-hand side of Eq. (30) contains the initial kinetic energy and its viscous dissipation while the right-hand side contains potential energy and its viscous dissipation. Both sides must not be negative.

3 Discussion

The potential energy dissipation coefficient \(C_s\) and kinetic energy dissipation coefficient \(C_K\) in Eq. (30) have yet to be determined. As discussed earlier, \(C_s\) shows the viscous effect during spontaneous spreading and can be numerically or experimentally determined based on the result of maximum spontaneous spread \(D_s\) (Eq. (17)). For \(C_K\), we first consider two particular cases: \(\theta_e=180\) deg and 0 deg. When \(\theta_e=180\) deg, \(H_s=0\). From Eq. (26), \(C_K=0\) for \(\theta_e=180\) deg. For \(\theta_e=0\) deg, we know that \(D_s=D_m=0\), \(C_s=1\), and \(H_s=H_m=0\). Hence, from Eq. (30), \(C_K=1\) for \(\theta_e=0\) deg. However, the determination of \(C_K\) is complicated for 0 deg < \(\theta_e<180\) deg. Extensive numerical or experimental efforts will be needed to correlate \(C_K\) with \(\theta_e\), \(W e\), and \(Re\), which will be the next step following the present work. In order to examine Eq. (30), we intend to choose an arbitrary correlation for \(C_K\) based on our basic understanding of physics. When the viscous effect is highly significant (i.e., \(Re \rightarrow 0\)), \(C_K \rightarrow 1\). This is because most of the initial kinetic energy could be dissipated when the droplet reaches the maximum spontaneous spread. To solely satisfy this criterion, we can arbitrarily define a correlation of \(C_K\) with \(Re\) in the form
\[
C_K = \frac{1}{1 + \sqrt{Re}}
\]

for 0 deg < \(\theta_e<180\) deg. We choose \(\sqrt{Re}\) in Eq. (31) because it also appears in the second term on the right-hand side of Eq. (28). Hence, \(C_K(\theta_e=0) = 1\), \(C_K(\theta_e=180) = 0\) together with Eq. (31) are used to plot Eq. (30) for further discussion.

Figure 5 shows \(D_m\) as a function of \(W e\) for three equilibrium contact angles \(\theta_e=30\) deg, 90 deg, and 150 deg, two values of Ohnesorge numbers \(Oh=1\) and 5, and the potential energy dissipation coefficient \(C_s=1\). It shows that \(D_m\) increases with \(W e\), showing that the initial kinetic energy causes the increase in maximum spread. The maximum spreads for \(Oh=1\) (less viscous) are larger than those for \(Oh=5\) (more viscous). Droplets with smaller equilibrium contact angles (higher wettability) have larger maximum spreads than those with larger equilibrium contact angles (lower wettability). The maximum spreads \(D_m\) for \(W e=0\) are the equilibrium spreads determined by \(\theta_e\) since \(C_s\) is set at unity.

Figure 6 shows \(D_m\) as a function of \(W e\) for two equilibrium contact angles \(\theta_e=30\) deg and 150 deg, three values of potential

Fig. 5 New model of maximum spread as a function of Weber number (\(C_s=1\))
energy dissipation coefficient $C_S=0$, 0.5 and 1, and Oh=1. For $\theta_e=30^\circ$, $D_m$ is larger for $C_S=0$ (inviscid spontaneous spreading) than that for $C_S=1$ (highly viscous spontaneous spreading). However, $D_m$ for $\theta_e=180^\circ$ is independent of $C_S$ since there is no spontaneous spreading for this nonwetting condition. The values of $D_m$ for $\theta_e=30^\circ$ and We=0 are the maximum spontaneous spread determined by $C_S$ while $D_m$ is zero for $\theta_e=180^\circ$ when We=0 (see the small figure embedded in Fig. 6).

Figure 7 shows $D_m$ as a function of $\theta_e$ (0 deg $< \theta_e \leq 180$ deg) for three values of Weber number We=0 (spontaneous spreading), 1, and 5, $C_S=0$, 0.5 and 1, and Oh=1. Figure 7 shows several important trends. First, $D_m$ increases with a decrease in $\theta_e$. Second, $D_m \to \infty$ when $\theta_e \to 0$ deg. Third, droplets with initial kinetic energy (We > 0) have larger maximum spreads than spontaneous spreading drops (We=0). Fourth, droplets with less viscous dissipation in spontaneous spreading (e.g., $C_S=0$) tend to have larger maximum spreads.

The difference between the present model and previous models is obvious by comparing Figs. 6 and 7 to Fig. 1. Previous models predict limited spreads for complete wetting and they show maximum spreads decreasing with an increase in Weber number. All these wrong tendencies have been corrected in the present model.

4 Conclusions

This paper presents a new analytical model, which is capable of predicting the maximum spread of single droplets on solid surfaces with zero or low velocities. The droplet is assumed to maintain a spherical cap shape during spreading, which is a reasonable geometric assumption for low velocity droplet impact. The model is based on energy conservation considering the droplet as a control volume. The change in potential energy is analyzed by considering the work done by the liquid surface tension force and the surface tension forces from the solid surface. The spreading time is approximated as the change in droplet height divided by droplet impact velocity. The viscous dissipation during spontaneous spreading and the viscous dissipation of the initial kinetic energy are considered separately and two coefficients are introduced: potential energy dissipation coefficient $C_S$ and kinetic energy dissipation coefficient $C_K$.

References